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ORO-T-323

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A Stochastic Analysis of Lanchester's Theory of Combat (U)

by

Richard H. Brown

ORO
OPERATIONS RESEARCH OFFICE
The Johns Hopkins University

Operating Under Contract with the DEPARTMENT OF THE ARMY

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FOR THE CHIEF OF RESEARCH AND DEVELOPMENT:

Herbert W. Mansfield

HERBERT W. MANSFIELD
Colonel, GS
Chief, Operations and Personnel Research Division

1 Incl
ORO-T-323

WORKING PAPER

This is a working paper of a consultant to the Strategic Division concerned with ORO Study No. 31.4.

It is the objective of the study to present a treatment of stochastic attrition processes by means of differential equations, difference equations, functional equations, and similar mathematical devices. This paper (ORO-T-323) deals with one aspect of the over-all objective. The findings and analysis of this paper are subject to revision as may be required by new facts or by modification of basic assumptions. Comments and criticism of the contents are invited. Remarks should be addressed to:

The Director
Operations Research Office
The Johns Hopkins University
7100 Connecticut Avenue
Chevy Chase 15, Maryland

STRATEGIC DIVISION
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Richard H. Brown

OPERATIONS RESEARCH OFFICE
The Johns Hopkins University Chevy Chase, Maryland

CONTENTS

PROBLEM

To investigate certain properties of stochastic attrition processes—specifically to formulate the general problem of the interaction of two groups of combatants when chance plays a role, to show what data must be given before the problem can be solved, to determine explicitly a complete solution of the general problem, to indicate a method for surmounting the practical difficulties encountered, and to carry out this method by analyzing the probability that a given side will win.

FACTS

F. W. Lanchester published[1] an analysis of the effects of concentration of firepower in combat. He considered two groups of forces opposing one another, assuming that the rate of attrition for each force depended on the strength of the opposition. He thus obtained a system of differential equations describing the behavior of the two forces.

During World War II there was a revival of interest in this theory of combat, though Lanchester's strictly deterministic treatment of the problem was modified to include the random fluctuations, which inevitably occur. Fundamental work in formulation of the problem in a stocnastic framework was carried out by the Operations Research Group in the Department of the Navy; such formulations are now being used at ORO in combat models.

DISCUSSION

In applying a stochastic analysis to Lanchester's theory of combat this memorandum considers a certain Markov stochastic process characterized by the initial state and the two functions ϕ and α. The probability that the system remains in a given state (m,n) throughout a time interval of length t is $e^{-\phi(m,n)}$. Alternatively it is $e^{-t/\tau(m,n)}$, where $\tau(m,n)$ is the expected duration of the system in the state (m,n) and $\tau(m,n) = 1/\phi(m,n)$. The probability that, leaving the state (m,n), the system goes to the state $(m,n-1)$ is $\alpha(m,n)$; the probability that, leaving the state (m,n), it goes to the state $(m-1,n)$ is $1-\alpha(m,n)$, which is also called $\beta(m,n)$. To find $p(a,b,t;m,n)$ is to determine the probability that, starting at (m,n), the system will be at (a,b) after the elapsed time t. Winning means the complete annihilation of the opposing side; the probability that the first side wins from state (m,n) is denoted by $P(m,n)$. This probability satisfies the difference equation $P(m,n) = \alpha(m,n)\ P(m,n-1) + \beta(m,n)\ P(m-1,n)$.

SUMMARY

CONCLUSIONS

1. The initial state, the expected duration in each possible subsequent state, and the chances of the transition from each state to the two possible immediately succeeding states completely determine the stochastic process.

2. Exact expression for the probability that a given side wins is too complicated to be of practical use, but it is demonstrated that it is possible to find a simple useful approximation.

3. The two functions ϕ and α, which characterize the stochastic process, may be expressed in terms of more readily accessible data, namely, the rate of fire and the single-shot kill probability, using the basic assumptions from Lanchester's original work.

A STOCHASTIC ANALYSIS OF LANCHESTER'S THEORY OF COMBAT

INTRODUCTION

An analysis of the effects of concentration of firepower in combat was made by F. W. Lanchester in 1916. He presented a system of differential equations determining the behavior of two groups of forces opposing one another, assuming for each that the rate of attrition depended on the strength of the opposition. Lanchester's theory of combat was deterministic; his differential equations implied that the future was completely determined. During World War II, however, a more realistic approach to the problem was undertaken, and a theory allowing for chance fluctuations was created. It is this stochastic theory which is studied here.

In this paper an attempt is made to perform three services for persons working in this field at the present time:

(a) To provide a general formulation of the problem, in terms of which the various specialized results of other groups of investigators can be interpreted. This is done in the first section.

(b) To exhibit a complete solution of the general problem as thus formulated. This is done in the second section. Although the solution is complete it is, in its general form, of little practical use. In various special cases, however, this solution is useful in studying certain properties of the solution. These will readily occur to the worker in the field.

(c) To indicate a method by which the practical difficulties encountered in studying the general solution can be surmounted. This is done in the third section, largely by exhibiting the method as applied to a specific aspect of the general problem. Although the problem studied in the third section is quite special, it seems clear that the asymptotic methods used there can be extended to other problems arising from the solution given in the second section. Work along these lines is now in progress.

To obtain the desired generality the formulation of the problem in the first section is necessarily abstract. It is proposed to give here a concrete formulation of the problem so that some meaning can be attached to the ideas in the first section.

Consider two opposing forces called the "first side" and the "second side." The forces try to destroy one another; hence, as time goes on, the strengths of the two sides diminish. Let $x(t)$ denote the number of survivors on the first side at time t, and let $y(t)$ denote the number of survivors on the second side at time t. The fundamental point of view in this paper is that $x(t)$ and $y(t)$ are random variables: they are not completely determined beforehand, but the probability that $x(t)$ and $y(t)$ have given values is. Specifically, the question is "What is the

probability $p(a,b,t;\ m,n)$ that there will be a and b survivors on the first and second sides, respectively, at time t, if there were initially m and n members on the two sides?"

The first section is concerned with the precise formulation of this problem, in addition to a discussion of the parameters that serve to characterize the effectiveness of each side in combat. For completeness the differential-difference equations for the problem, used by many workers in stochastic processes, are also derived. The asymptotic devices used in the third section, however, are not based on these equations.

In the second section $p(a,b,t;\ m,n)$ is completely determined. It is exhibited however as an integral in which the complicated structure of the integrand tends to obscure the understanding of the precise nature of the solution. This complexity arises mainly from certain combinatorial difficulties associated with "paths," defined in the first section and examined more fully in the second. The situation is somewhat analogous to the situation in certain one-dimensional stochastic processes, where the exact solution is apparently quite useless.

If a useful solution to the problem of finding $p(a,b,t;\ m,n)$ cannot be given, then an effort can be made to answer related questions. One question of evident interest is this: "What is the probability that the first side will win?" Here winning means the destruction of the opposing force before a given side is annihilated. Again, a precise answer is complicated beyond usefulness, but a useful approximate answer can be given. In the third section this approximate answer is found. Comparison of the exact solution and the approximation in certain numerical examples shows that the agreement is surprisingly good. These examples are given in the last section.

It must be admitted that many problems remain to be solved before a coherent stochastic theory of combat is evolved. This memorandum claims only to frame the problem in a precise way, then to formulate an exact solution of this problem (to spare future workers the labor of writing out this solution), and finally to present a useful approximation to the probability that one side will win. Among the many other problems that remain, the three following are of special interest:

(a) What is a useful approximation for $p(a,b,t;\ m,n)$?

(b) Find an approximation for the expected number of survivors on each side in terms of time. How close is this to the solution of the classical deterministic Lanchester equations?

(c) How, if at all, can the central limit theorems of Bernstein and Loève for <u>dependent</u> probabilities be used here to simplify the derivation of asymptotic formulas?

It is hoped that this memorandum may help to stimulate work on these related problems.

STATEMENT OF THE PROBLEM

Consider a physical system, the state of which at any instant t is described by an ordered pair of nonnegative integers $[x(t),y(t)]$. The system is to be observed, starting at some convenient time which will be taken as zero. The actual behavior of the system is thus completely prescribed by two functions, x and y, defined on the set of all nonnegative real numbers, with values in the set of nonnegative integers. The possible behavior of the system is circumscribed by the following two assumptions: first, that both x and y are nonincreasing functions, i.e., if $t'>t$, then $x(t') \leq x(t)$ and $y(t') \leq y(t)$; and, second, that when the system leaves any state (a,b), it must proceed to the state $(a-1,b)$ or the state $(a,b-1)$.

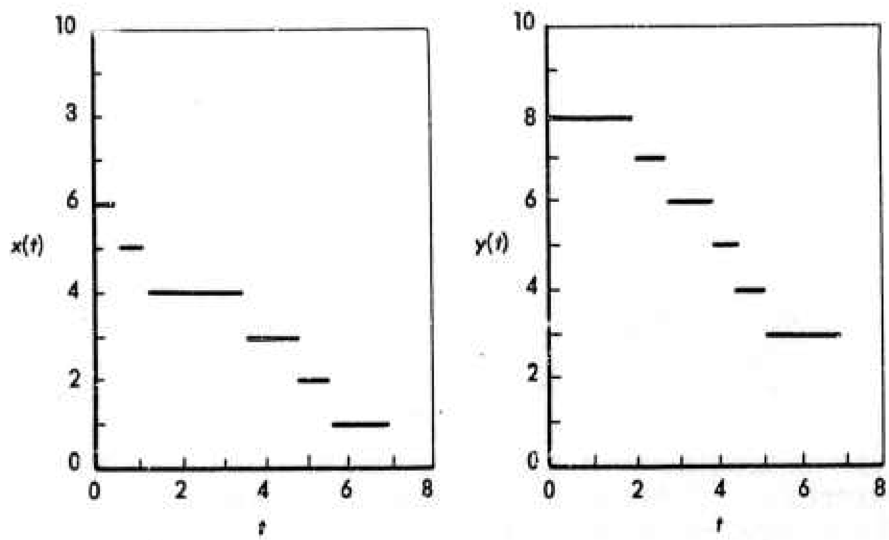

Fig. 1—Typical Graphs of $x(t)$, $y(t)$.

The set of all possible modes of behavior of this system may thus be identified with the set of all ordered pairs of nonincreasing, nonnegative, integral-valued functions, defined on the set of all nonnegative real numbers, such that both functions in the ordered pair are not discontinuous at the same number, and such that at a point where it is discontinuous, a function decreases by one unit. The set of all ordered pairs of such functions will be denoted by S. For a further description of the sort of situation envisioned here, the reader is referred to work by Koopman, cited in Morse and Kimball.[2]

An element of the set S is an ordered pair of functions. An example of the graphs of such an ordered pair of functions might look like Fig. 1.

An alternative (and not so informative) method of representing this behavior graphically is to show the path followed by the system, disregarding the time of transition from one state to another. The path diagram which would correspond to the situation shown in Fig. 1 is given in Fig. 2.

It should be evident that each element of S (i.e., each ordered pair of functions of the type being considered) determines uniquely a graph in the fashion of Fig. 2. On the other hand, to each graph of the type shown in Fig. 2, infinitely many elements of the set S will generally correspond (or, equivalently, infinitely many graphs like Fig. 1).

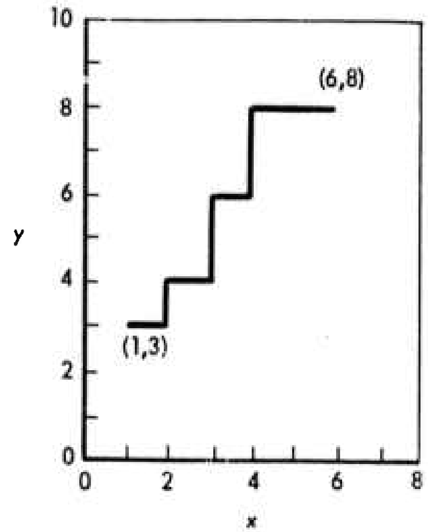

Fig. 2—The Path for Fig. 1.

Thus far the discussion has centered on the possible representation of the behavior of the system. It is, of course, a much more fundamental problem to discuss the mechanism leading to a particular mode of behavior for the system. It might be, for example, that the first coordinate of the system decreases by one unit at a quarter past and a quarter to every hour, but the second coordinate decreases by one unit every hour on the hour. In such a case, only the initial state of the system need be known in order that the state of the system be known at any subsequent time. Under these conditions the process is said to be deterministic. The process considered in Morse and Kimball[3] is deterministic.

This is just the situation that will not be considered here. Instead it will be assumed that chance affects the system, so that when the initial state of the system is known, only the probability that the system will be in a certain state at a subsequent time is known. Such a process is said to be "stochastic." Morse and Kimball also consider a stochastic process.[4] For a more detailed but more technical description of stochastic processes in general, the reader is referred to Doob.[5]

As a generalization of the situation in a great many practical problems the following assumptions are made for the stochastic process being considered here: first, that it is a Markov process, and, second, that it is a stationary process. For the technical definitions of a Markov process and a stationary process, see Doob.[6] Both terms will be used here "in the strict sense." An intuitive interpretation of these assumptions follows.

The assumption that the process is a Markov process can be expressed intuitively by saying that, from any instant onward, the behavior of the system may depend on the state of the system at that instant, but it certainly does not depend on the previous history of the system. Thus, if $0 \le t_1 < t_2 < \ldots < t_n < t$, the probability that the system is in state (a,b) at the instant t, given that it was in state (a_1,b_1) at t_1, (a_2,b_2) at t_2, . . ., and (a_n,b_n) at t_n, depends only on a, b, t, a_n, b_n, and t_n, not on the states at the earlier instants. Consequently only probabilities of the form $p(c,d,t;\ a,b,s)$ need be considered, which denote the probability that the system is in state (c,d) at the instant t, given that the system is in state (a,b) at the instant s, with $s < t$.

The second assumption (that the process is stationary) introduces a further simplification. It means that what happens during a given time interval depends only on the state of the system at the beginning of the interval, and on the length of the time interval—not on the instant at which the time interval begins. Consequently $p(c,d,s+t;\ a,b,s) = p(c,d,t;\ a,b,0)$, which might as well be (and which will be) denoted by $p(c,d,t;\ a,b)$.

In summary this investigation is to be concerned with the probabilities $p(c,d,t;\ a,b)$, which denote the probability that, if the system is at any instant in the state (a,b), then it will be in the state (c,d) after a time interval of length t.

Certain properties of $p(c,d,t;\ a,b)$, which are consequences of the preceding assumptions, will now be derived.

Suppose now that the system is initially in a state (m,n). The probability that it will proceed to a state (a,b) during an interval of time of length s is $p(a,b,s;\ m,n)$, and the probability that it will then proceed to a state (c,d) during a succeeding interval of length t is $p(c,d,t;\ a,b)$, since the system is in state (a,b) at the beginning of this second interval. Thus the probability that, if the system is initially at (m,n), it proceeds to (a,b) during the interval of length s and thence to (c,d) during the succeeding interval of length t is $p(a,b,s;\ m,n) \cdot p(c,d,t;\ a,b)$.

Consider next the probability that, if the system is initially at (m,n), it proceeds to (c,d) during an interval of length $s+t$. It is, of course, denoted by $p(c,d,s+t;\ m,n)$. But, this can also be expressed as a sum of products of the type just considered. For at the end of the first interval of length s the system must be in some state between (m,n) and (c,d), inclusive. Since it can be in only one state at a given instant, the various possibilities are mutually exclusive. Hence, $p(c,d,s+t;\ m,n)$ is the sum of all the probabilities of the compound events described in the preceding paragraph; i.e.

$$p(c,d,\ s+t;\ m,n) = \sum_{a=c}^{m} \sum_{b=d}^{n} p(a,b,s;\ m,n)\, p(c,d,t;\ a,b). \tag{1}$$

Equation 1, which will be found to be fundamental in the subsequent work, is essentially a special case of Feller's[7] Eq. 9.2 or of Doob's[8] Eq. 1.2′.

A special case of Eq. 1 arises if $c = m$ and $d = n$. In this case

$$p(m,n,\ s+t;\ m,n) = p(m,n,s;\ m,n)\, p(m,n,t;\ m,n).$$

[The appropriate verbal interpretation of $p(m,n,t;\ m,n)$ is the probability that the system remains in the state (m,n) throughout a time interval of length t, if the system is in this state as the beginning of the time interval.]

Because p is a probability function, it follows from the general theory of stochastic processes that

$$p(m,n,t;\ m,n) = e^{-t\phi(m,n)}, \tag{2}$$

where $\phi(m,n)$ is nonnegative. Indications of the method of proof of this result are found in Doob.[9]

If certain (unnecessary) assumptions are made about the function p, a simple proof of Eq. 2 can be given. The special case of Eq. 1 under consideration is an equation of the form

$$f(s+t) = f(s)\, f(t).$$

Differentiate with respect to s; then

$$f'(s+t) = f'(s)\, f(t).$$

Now set s equal to zero, obtaining

$$f'(t) = c\, f(t),$$

where $c = f'(0)$. This is a differential equation whose solutions are of the form

$$f(t) = ae^{ct},$$

where a is an arbitrary constant. Now in the special case of Eq. 1, which is of interest here, the condition that $p(m,n,0;\ m,n)$ should equal one requires that a be one, and the condition that $p(m,n,t;\ m,n)$ be less than one for positive t requires that c in the exponent be negative. This establishes Eq. 2.

Of course, this "proof" has assumed that all the functions encountered possess derivatives. This is an unnecessary assumption, as the work in Doob shows.

Because of the importance of the facts expressed by Eq. 2, the equation is given the following verbal interpretation: The probability that the system does not change from the state (m,n) during an interval of time of length t is $e^{-t\phi(m,n)}$.

Consider next the variate (or "random variable") $T_{m,n}$ whose value is the length of time during which the system remains in the state (m,n).

Let F denote the (cumulative) distribution function for $T_{m,n}$; i.e., $F(u)$ is the probability that the observed value of the variate $T_{m,n}$ is less than or equal to u. Then, for any positive number u,

$$\begin{aligned} F(u) &= Pr\{T_{m,n} \le u\} \\ &= Pr\{\text{the system is no longer in } (m,n) \text{ after a time } u\} \\ &= 1 - Pr\{\text{no change occurs during the time } u\} \\ &= 1 - e^{-u\phi(m,n)} \end{aligned}$$

while $F(u) = 0$ if u is zero, or negative. Obviously, $F'(u) = 0$, if $u < 0$, and $F'(u) = \phi(m,n)\, e^{-u\phi(m,n)}$, if $u > 0$.

The expected value of $T_{m,n}$, which will be denoted by $\tau(m,n)$, is given by the following procedure:

$$\tau(m,n) = \int_{-\infty}^{\infty} u\,dF(u) = \int_{0}^{\infty} u\phi(m,n)\, e^{-u\phi(m,n)}\, du = 1/\phi(m,n).$$

Thus $\phi(m,n)$ can be interpreted as the reciprocal of the average duration of the system in the state (m,n). Perhaps it should be stated explicitly that the symbol $\tau(m,n)$ will be used throughout the sequel to mean the expected value of $T_{m,n}$.

Up to this point the discussion appears to be concerned principally with the probability that the system does not undergo any changes of state during a given interval of time. The next part of the discussion is primarily concerned with establishing the notation necessary for considering what happens when the system does change its state.

Suppose the system is in a certain state (m,n). By the second of the fundamental assumptions made at the beginning of this section, the system can proceed only to states $(m-1,n)$ and $(m,n-1)$ when it leaves the state (m,n). Let $\alpha(m,n)$ denote the probability that the system, on leaving the state (m,n), proceeds to the state $(m,n-1)$, and let $\beta(m,n)$ denote the probability that the system, on leaving the state (m,n), proceeds to the state $(m-1,n)$. Of course, $\beta(m,n) = 1-\alpha(m,n)$.

It may now be intuitively evident that if the initial state of the system (corresponding to $t=0$), and the two functions ϕ and α are given, then all the probabilities $p(a,b,t;\ m,n)$ are completely determined. In any event, this is the case, as the following discussion will show.

The principal objective now is to derive two differential-difference equations that must be satisfied by $p(a,b,t;\ m,n)$. Each such equation is based on Eq. 1.

First Equation. Differentiate Eq. 1 with respect to s, and then set s equal to zero. The result is

$$p_3(c,d,t;\ m,n) = \sum_{a=c}^{m} \sum_{b=d}^{n} p_3(a,b,0;\ m,n)\ p(c,d,t;\ a,b),$$

in which the subscript 3 denotes the first-order partial derivative with respect to the third argument; i.e.,

$$p_3(c,d,t;\ m,n) = \lim_{h\to 0}\ [p(c,d,\ t+h;\ m,n) - p(c,d,t;\ m,n)]/h.$$

Second Equation. Differentiate Eq. 1 with respect to t, and then set t equal to zero. The result is

$$p_3(c,d,s;\ m,n) = \sum_{a=c}^{m} \sum_{b=d}^{n} p(a,b,s;\ m,n)\ p_3(c,d,0;\ a,b).$$

Now it turns out that most of the terms that appear in the sums in these two equations are zero. It is clear from Eq. 2 that $p_3(a,b,0;\ a,b) = -\phi(a,b)$. It can be shown by the general theory, or directly by quite simple arguments, that

$$p_3(a,\ b-1,\ 0;\ a,b) = \alpha(a,b)\ \phi(a,b)$$

$$p_3(a-1,\ b,\ 0;\ a,b) = \beta(a,b)\ \phi(a,b),$$

and that all the other derivatives are zero. The general theory is given in Doob;[10] an alternative treatment, with less generality, is contained in Feller.[11]

If, in the second equation, the variable s is replaced by t, the two equations can be written now as

$$(\partial/\partial t)\ p(a,b,t;\ m,n) = -\phi(m,n)\ p(a,b,t;\ m,n) + \alpha(m,n)\ \phi(m,n)\ p(a,b,t;\ m,\ n-1) + \beta(m,n)\ \phi(m,n)\ p(a,b,t;\ m-1,\ n) \tag{3}$$

and

$$(\partial/\partial t)\ p(a,b,t;\ m,n) = -\phi(a,b)\ p(a,b,t;\ m,n) + \alpha(a,\ b+1)\ \phi(a,\ b+1)\ p(a,\ b+1,\ t;\ m,n) + \beta(a+1,\ b)\ \phi(a+1,\ b)\ p(a+1,\ b,t;\ m,n). \tag{4}$$

Clearly the solutions of these equations will depend only on the coefficients, which are determined by ϕ and α (since $\beta = 1 - \alpha$), and on the initial state (m,n). Hence, the validity of the earlier assertion regarding the dependence of $p(a,b,t;\, m,n)$ on ϕ and α has been established.

Equations 3 and 4 correspond, respectively, to Eqs. 1.7 and 1.7′ in Doob.[12] Such equations are frequently distinguished by the adjectives "backward" and "forward." Equation 3 is the backward one for this process. For a further discussion of these two equations, consult Doob,[13] or Feller.[14] Note also that an analogue of Eq. 3 appears in Morse and Kimball.[15]

It is now perhaps appropriate to summarize what has developed so far.

A certain stationary Markov stochastic process is under consideration. The process is characterized by the initial state, and the two functions, ϕ and α. The probability that the system remains in a given state (m,n) throughout a time interval of length t is $e^{-t\phi(m,n)}$. Alternatively, it is $e^{-t/\tau(m,n)}$, where $\tau(m,n)$ is the expected duration of the system in the state (m,n), and $\tau(m,n) = 1/\phi(m,n)$. The probability that, on leaving the state (m,n), the system goes to the state $(m,n-1)$ is $\alpha(m,n)$; the probability that, on leaving the state (m,n), it goes to the state $(m-1,n)$ is $1-\alpha(m,n)$, which is also called $\beta(m,n)$.

Finally, if $p(a,b,t;\, m,n)$ denotes the probability that the system, starting at state (m,n), is in state (a,b) after an elapse of time t, then p must satisfy both Eqs. 3 and 4.

The fundamental problem, which will be solved completely in the next section can now be stated:

Given the two functions ϕ and α, write out explicitly $p(a,b,t;\, m,n)$.

EXACT DETERMINATION OF THE PROBABILITIES $p(a,b,t;\ m,n)$

In this section $p(a,b,t;\ m,n)$ will be determined exactly. However, it will be seen that the result, unless a is close to m, and b is close to n, is of little immediate practical interest in the general case.

An obvious possible way to determine $p(a,b,t;\ m,n)$ is to solve the partial differential-difference equations, Eqs. 3 and 4 of the preceding section. It is the author's experience, however, that this method does not lead—in any evident way—to a form of the solution that is convenient for the applications that have given rise to this research. Consequently the problem will be attacked by methods from the general theory of probability.

To find $p(a,b,t;\ m,n)$ is to answer the question, "What is the probability that, starting at (m,n), the system will be at (a,b) after an elapsed time t?"

For the system to start at (m,n) and later be at (a,b), it must proceed along one of the paths (as in Fig. 2) that join (m,n) to (a,b). It may move along any such path, and motions along different paths are mutually exclusive events.

It is easy to compute the probability that the system will move along a given path. For example, suppose the system, initially at (5,7) proceeds through (5,6), (5,5), (4,5), (4,4), (4,3), and (3,3) to (3,2). The probability of going from (5,7) to (5,6) is $\alpha(5,7)$; from (5,6) to (5,5) is $\alpha(5,6)$; from (5,5) to (4,5) is $\beta(5,5)$; etc. Hence the probability of moving along this particular path is $\alpha(5,7) \cdot \alpha(5,6) \cdot \beta(5,5) \cdot \alpha(4,5) \cdot \alpha(4,4) \cdot \beta(4,3) \cdot \alpha(3,3)$.

Similarly, if there is a quite arbitrary path from (m,n) to (a,b), there would be a certain α for each downward step and a certain β for each leftward step. The product of these α's and β's would give the probability of moving along this particular path.

In order to present these ideas as clearly as possible and to facilitate certain future discussions it seems desirable to introduce the following method for describing paths:

Consider a path from (m,n) to (a,b). Each transition consists of a motion either one step to the left (the first coordinate decreases by one unit) or one step down (the second coordinate decreases by one unit). A convenient way to describe the path is to write down a sequence of $m-a$ zeros and $n-b$ ones, a zero corresponding to a motion to the left, and a one corresponding to a motion down. In this system, the path described a little earlier, beginning at (5,7) and ending at (3,2), would be written 1101101.

To formalize this procedure in the general case, proceed as follows: Let $x = m - a$ and let $y = n - b$. Consider the set, $I_{x,y}$, of all positive integers, which in their usual binary representation contain exactly y ones and no more than x zeros. If, in the usual representation, the integer contains y ones and $x - c$ zeros,

let c zeros precede the usual representation. This will be called the "modified binary representation." For instance, if $x = 4$ and $y = 2$, the integer 6 usually represented by 110 would be written 000110 in the modified representation. If k is any integer in $I_{x,y}$, and if the modified representation of k is $i_1 i_2 \ldots i_{x+y}$, define $\delta_{k,j}$ as i_j. Thus, if $x = 1$ and $y = 2$, one would have $\delta_{51} = \delta_{53} = 1$ and $\delta_{52} = 0$, since $5 = 101$.

Now there is, in the fashion previously described, a one-to-one correspondence between the integers in the set $I_{x,y}$ and the paths connecting (m,n) to (a,b). Let π_k denote the path that corresponds to the integer k in $I_{x,y}$. Then $\delta_{k,j} = 0$ means that the jth transition for π_k is to the left, and $\delta_{kj} = 1$ means that the jth transition is downward.

Notice that if $\delta_{k1} = 0$, the second state is $(m-1,n)$; if $\delta_{k1} = 1$, the second state is $(m, n-1)$. In either case, the second state is $(m-1+\delta_{k1}, n-\delta_{k1})$. In general, as can easily be seen, the system will be in state

$$(m - r + \sum_{j=1}^{r} \delta_{kj},\ n - \sum_{j=1}^{r} \delta_{kj})$$

after r transitions, provided that $r \leq x + y$.

In this way the notation that has just been introduced enables one to talk freely, but specifically, about the motion of the system.

An example of the use of this notation is provided in writing down the probability, say $p_k(a,b;\ m,n)$, that the system proceeds from the state (m,n) to the state (a,b) along the path π_k. It is known from the earlier discussion that $p_k(a,b;\ m,n)$ is a product of certain α's and β's, which are completely determined by the path, but in ordinary terms it is hard to state just which α's and β's are to be used.

Consider the path π_k. Define $\gamma_{k,1}$ as $\alpha(m,n)$ if $\delta_{k1} = 1$, and as $\beta(m,n)$ if $\delta_{k1} = 0$. In general, define $\gamma_{k,\,r+1}$, for $r = 0, 1, \ldots, x+y-1$ as

$$\alpha(m - r + \sum_{j=1}^{r} \delta_{kj},\ n - \sum_{j=1}^{r} \delta_{kj}) \text{ if } \delta_{k,r+1} = 1,$$

and as

$$\beta(m - r + \sum_{j=1}^{r} \delta_{kj},\ n - \sum_{j=1}^{r} \delta_{kj}) \text{ if } \delta_{k,r+1} = 0.$$

Specifically,

$$\gamma_{k,r+1} = \delta_{k,r+1}\,\alpha(m - r + \sum_{j=1}^{r} \delta_{kj},\ n - \sum_{j=1}^{r} \delta_{kj}) + (1 - \delta_{k,r+1})\,\beta(m - r + \sum_{j=1}^{r} \delta_{kj},\ n - \sum_{j=1}^{r} \delta_{kj}).$$

Now,

$$p_k(a,b;\ m,n) = \prod_{j=1}^{x+y} \gamma_{k,j}.$$

A summary of the results up to this point of the discussion might indicate that a notation has been introduced that facilitates writing down the probability that the system, starting at (m,n), travels a certain path to (a,b).

Of course it is necessary to know more, e.g., the probability, say $G_k(a,b,t;\ m,n)$, that, if the system travels the path π_k from (m,n) to (a,b), it will be at (a,b) at a certain time t. If $G_k(a,b,t;\ m,n)$ were known, $p(a,b,t;\ m,n)$ would also be known, since

$$p(a,b,t;\ m,n) = \sum_{k \in I_{x,y}} p_k(a,b;\ m,n)\ G_k(a,b,t;\ m,n).$$

Perhaps the simplest way to find $G_k(a, b, t;\ m,n)$ is to use the variates $T_{c,d}$ discussed in the preceding section, dealing with the statement of the problem. If it is known that the system moves along the path π_k, then it will be at (a,b) at time t if and only if $T_{m,n} + \ldots + T_{ab} > t$, but the sum of all these variates except the last is $\leq t$. The variates appearing in the two sums are to be those arising along the particular path π_k considered.

It is possible to be more specific about the variates present in these sums if the previous notation for the description of the path is used.

The state in which the system will be found after r transitions as it proceeds along π_k was written out earlier. If this state is denoted by $[m(k,r), n(k,r)]$, the earlier result was that

$$m(k,r) = m - r + \sum_{j=1}^{r} \delta_{kj}$$

and

$$n(k,r) = n - \sum_{j=1}^{r} \delta_{kj} \ .$$

The variates that must be considered on π_k are $T_{m(k,r),\,n(k,r)}$, for $r = 0, 1, \ldots, x+y$. The probability $G_k(a,b,t;\ m,n)$, then, is the probability of the compound event

$$\sum_{r=0}^{x+y-1} T_{m(k,r),\ n(k,r)} \leq t \text{ and } \sum_{r=0}^{x+y} T_{m(k,r),\ n(k,r)} > t \ .$$

The probabilities that enter into the computation of the probability of this compound event can be found quite directly by the ordinary method of characteristic functions.

Let f_{cd} denote the characteristic function for the variate T_{cd}; then

$$f_{cd}(u) = \int_0^\infty e^{ius} \phi(c,d)\, e^{-s\phi(c,d)}\, ds = \frac{\phi(c,d)}{\phi(c,d) - iu} = \frac{1}{1 - iu\, r(c,d)} .$$

Now the variates $T_{m,n}, \ldots, T_{a,b}$ are independent, since the process is a stationary Markov process. Consequently the characteristic function for the sum

$$\sum_{r=0}^{x+y-1} T_{m(k,r),\ n(k,r)}$$

is the product

$$\prod_{r=0}^{x+y-1} f_{m(k,r),\ n(k,r)} \ ,$$

and the density function, $g_k(s)$, for this sum is determined by the equation

$$g_k(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ius} \prod_{r=0}^{x+y-1} \frac{1}{1 - iu\, r[m(k,r),\ n(k,r)]}\, du .$$

The probability that $T_{a,b} > t - s$ is $e^{-(t-s)\,\phi(a,b)}$. Since

$$G_k(a,b,t;\ m,n) = \int_0^t g_k(s)\ e^{-(t-s)\,\phi(a,b)}\, ds,$$

it is found that

$$G_k(a,b,t;m,n) = \int_0^t e^{-(t-s)\,\phi(a,b)} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ius} \prod_{r=0}^{x+y-1} \frac{1}{1 - iu\,r[m(k,r),\, n(k,r)]}\, du \right\} ds.$$

This expression can be simplified in two ways. First, for neater notation, write m_{kr} and n_{kr} instead of $m(k,r)$ and $n(k,r)$. Second, interchange the order of integration, integrating first with respect to s, and then with respect to u. The integral involving s can be evaluated easily; it is found that

$$G_k(a,b,t;m,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{r=0}^{x+y-1} \frac{1}{1 - iu\,r(m_{kr},n_{kr})} \, \frac{e^{-iut} - e^{-t\phi(a,b)}}{\phi(a,b) - iu}\, du.$$

To complete the explicit determination of $p(a,b,t;\ m,n)$, merely combine this result with the earlier result concerning $p_k(a,b;\ m,n)$. The combination of the two expressions is rendered more compact by writing

$$\prod_{j=1}^{x+y} \gamma_{kj} = \prod_{r=0}^{x+y-1} \gamma_{k,r+1}\,.$$

The final result is that $p(a,b,t;\ m,n)$ is given by

$$\frac{1}{2\pi} \sum_{k \epsilon I_{x,y}} \int_{-\infty}^{\infty} \prod_{r=0}^{x+y-1} \frac{\gamma_{k,r+1}}{1 - iu\,r(m_{kr},n_{kr})} \, \frac{e^{-iut} - e^{-t\phi(a,b)}}{\phi(a,b) - iu}\, du.$$

The validity of the criticism made against this result at the beginning of this section should now be evident. If $x + y$ is at all large, the integrand can be [for general $r\ (m_{kr},n_{kr})$] quite complicated, and it is difficult to apprehend the nature of the solution.

THE PROBABILITY OF WINNING

In the introduction to this memorandum it was pointed out that the problem under investigation arose from a definite problem in the theory of combat. The theoretical development in the previous sections ignored completely any concrete interpretation of the results; the work was purely mathematical in character.

This section is devoted to the study of a problem arising from the combat interpretation of the stochastic process described in the first two sections. If the stochastic process is considered as describing the interaction of two groups of combatants, then one question of quite great interest arises: "What is the probability that a given side will win?"

Now, of course, the concept "winning" must be clarified. A simple definition (and the one that will be used here) is that winning means the complete annihilation of the opposing side. The symbol $P(m,n)$ will denote the probability that the first side wins, if there are initially m combatants on the first side, and n combatants on the second side.

Relevant notation from the previous sections will be preserved here. In particular, $\alpha(m,n)$ will denote the probability that, if there are m combatants on the first side and n combatants on the second side, then the next casualty to occur will be on the second side, and $\beta(m,n)$ will be $1-\alpha(m,n)$. More briefly, $\alpha(m,n)$ is the probability of having the system proceed from (m,n) to $(m,n-1)$, and $\beta(m,n)$ is the probability of having the system proceed from (m,n) to $(m-1,n)$.

If the system is in state (m,n), there are two mutually exclusive ways in which the first side can win. The system might proceed to the state $(m,n-1)$, with the first side then winning. The probability of this is $\alpha(m,n)\,P(m,n-1)$. Or, the system might proceed to the state $(m-1,n)$ with the first side then winning. The probability of this is $\beta(m,n)\,P(m-1,n)$. Since these are mutually exclusive events, the probability that the first side wins from the state (m,n) is the sum of these two probabilities. The following difference equation is therefore obtained.

$$P(m,n) = \alpha(m,n)\,P(m,n-1) + \beta(m,n)\,P(m-1,n). \tag{5}$$

Equation 5 is the fundamental equation for the work that follows. It is of some interest to observe that it is a consequence of Eq. 3. It may be obtained from Eq. 3 by setting $b=0$, summing for $a=1,2,\ldots,m$, and letting t become infinite. The partial derivatives on the left all approach zero as t becomes infinite. Equation 21, which appears later, is similarly related to Eq. 4.

A brief reflection will show that if $P(m,0)$ is known for $m=1,2,\ldots,$ and if $P(0,n)$ is known for $n=1,2,\ldots,$ then Eq. 5 completely determines $P(m,n)$ at

all lattice points in the first quadrant (i.e., at all points having integral coordinates). On the other hand, from the definition of $P(m,n)$, it is obvious that

$$P(m,0) = 1 \text{ for } m = 1, 2, \ldots \qquad (6)$$
$$P(0,n) = 0 \text{ for } n = 1, 2, \ldots .$$

These equations will occasionally be referred to as the "boundary conditions."

In principle $P(m,n)$ could now be written out explicitly by solving Eq. 5. If the coefficients $\alpha(m,n)$ are not too complicated, this is, in fact, possible. For example, suppose that $\alpha(m,n)$ is a constant, say α. Then

$$P(m,n) = \sum_{k=0}^{m-1} \frac{(n+k-1)!}{(n-1)!\,k!} \alpha^n (1-\alpha)^k . \qquad (7)$$

Or suppose that

$$\alpha(m,n) = \frac{\alpha m}{\alpha m + \beta n},$$

where α and β are constants. Then it can be shown that

$$P(m,n) = \left(\frac{\alpha}{\beta}\right)^n \sum_{k=1}^{m} (-1)^{m-k} \frac{k^{m+n}\,\Gamma[(\alpha k/\beta)+1]}{(m-k)!\,k!\,\Gamma[n+(\alpha k/\beta)+1]} . \qquad (8)$$

The simplest way of verifying the truth of these two assertions is to observe that the given function satisfies both Eqs. 5 and 6, and then appeal to the obvious uniqueness of the solution.

Now if m and n are at all large, both Eqs. 7 and 8 offer difficulties from the point of view of computation as well as from the point of view of simple understanding of the nature of the solution. Indeed Eq. 8 is especially bad, since it involves the differences of numbers that may be quite large.

For these two reasons the project of obtaining the exact solution of Eq. 5 will be abandoned. Rather the problem to be considered here is that of finding a simple approximation to $P(m,n)$, which will be valid if $m+n$ is large.

It should be intuitively evident that without some restriction on the behavior of the coefficients $\alpha(m,n)$, general results on the asymptotic behavior of $P(m,n)$ cannot be obtained. Fortunately the very nature of the combat situations being investigated here imposes just such a restriction, which will now be stated.

Consider the two numbers $\alpha(m,n)$ and $\alpha(m+1,n)$. These represent the probabilities of a casualty on the second side when the first side has strengths of m or $m+1$. It is certainly reasonable to assume that the stronger the first side, the greater the probability of a casualty on the second side; i.e., to assume that $\alpha(m+1,n) \geq \alpha(m,n)$. A similar discussion shows the reasonableness of assuming that $\alpha(m,n) \geq \alpha(m,n-1)$.

What has just been found to be reasonable will now be made into a formal assumption:

Throughout the rest of this section it will be assumed that

$$\alpha(m+1,n) \geq \alpha(m,n) \qquad (9)$$
$$\alpha(m,n) \geq \alpha(m,n-1).$$

It may be noted that all the particular forms of $\alpha(m,n)$ that the author has seen in the literature on Lanchester's theory satisfy these assumptions.

The process for finding an approximation to $P(m,n)$ is based on the observation that the value of P at any point on the straight line whose equation is $m+n=N$ is a linear combination of the values of P at certain points on the straight line whose equation is $m+n=N-1$. As the problem now stands $\alpha(m,n)$ is defined only if m and n are not both zero and neither is negative. To facilitate the language in the following discussion it is desirable to extend the domain of definition of α. From now on it will be assumed that $\alpha(m,n)$ is defined for all m,n (positive, negative, or zero) for which $m+n \geq 1$. For negative values of the arguments the following conventions are to hold:

$\alpha(m,n)=\alpha(m,0)$ for $n<0$;
$\alpha(m,n)=\alpha(0,n)$ for $m<0$;
$\beta(m,n)=1-\alpha(m,n)$ for all m,n.

It might be noted that this convention regarding values of $\alpha(m,n)$ in the fourth and second quadrants preserves the fundamental assumption of Eq. 9. It might also be noted that this extension of $\alpha(m,n)$ to the fourth and second quadrants is not essential to the arguments that follow; rather it simply serves to eliminate certain unimportant details that would otherwise have to be discussed.

It is now instructive to consider the values of both $P(m,n)$ and $\alpha(m,n)$ on the straight line for which $m+n=k$. (Although the functions are defined only for the lattice points, values can be interpolated for both functions so that the following observations are valid.)

First, $\alpha(m,k-m)$ is a monotone nondecreasing function of m, and $0 \leq \alpha(m,k-m) \leq 1$.

Second, $P(m,k-m)$ is a distribution function; i.e.,

i) $P(m,k-m)$ is a nondecreasing function of m
ii) $\lim\limits_{m\to-\infty} P(m,k-m)=0$
iii) $\lim\limits_{m\to\infty} P(m,k-m)=1.$

This suggests the possibility of trying to approximate $P(m,n)$ by a standard distribution function—for instance, the normal distribution function. For each value of k and m there certainly exists a number $W_k(m)$ such that

$$P(m,k-m)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{W_k(m)} e^{-\frac{1}{2}t^2}\,dt;$$

however, the precise form of $W_k(m)$ is irrelevant. It must be remembered that only an asymptotic formula is being sought.

Consider what would happen if $\alpha(m,n)$ were a constant, α. The classical theorem of De Moivre and Laplace regarding the approximation of the Bernoulli distribution by the normal distribution would show that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{am-bn}{\sqrt{m+n}}} e^{-\frac{1}{2}t^2}\,dt,$$

for a proper choice of a and b, is asymptotically equivalent to $P(m,n)$.

It is certainly reasonable to ask whether a similar approximation might not also hold in the problem considered here for general coefficients $\alpha(m,n)$.

At any rate it suggests the plausibility of the following change of variables, and that, in fact, is all that the preceding discussion was aimed at. To repeat: The following change of variables is not capricious; it is suggested by standard classical theorems.

Let

$$N = m + n$$
$$u = (am - bn)/(\sqrt{m + n}), \tag{10}$$

where both a and b are positive numbers to be determined later. Because a and b are positive, the transformation from (m,n) to (u,N) is nonsingular. The inverse transformation is

$$m = (bN + u\sqrt{N})/(a + b)$$
$$n = (aN - u\sqrt{N})/(a + b). \tag{11}$$

Let

$$P(m,n) = P_{m+n}[(am - bn)/(\sqrt{m + n})] = P_N(u)$$
$$\alpha(m,n) = \alpha_{m+n}[(am - bn)/(\sqrt{m + n})] = \alpha_N(u) \tag{12}$$
$$\beta_N(u) = 1 - \alpha_N(u).$$

Then

$$P(m,n - 1) = P_{m+n-1}[(am - bn + b)/(\sqrt{m + n - 1})] = P_{N-1}[u\sqrt{N/(N - 1)} + b/\sqrt{(N - 1)}]$$

and

$$P(m - 1,n) = P_{m+n-1}[(am - bn - a)/(\sqrt{m + n - 1})] = P_{N-1}[u\sqrt{N/(N - 1)} - a/\sqrt{(N - 1)}].$$

The fundamental difference equation (Eq. 5) now becomes

$$P_N(u) = \alpha_N(u)\, P_{N-1}[u\sqrt{N/(N - 1)} + b/\sqrt{(N - 1)}] + \beta_N(u)\, P_{N-1}[u\sqrt{N/(N - 1)} - a/\sqrt{(N - 1)}]. \tag{13}$$

This suggests the following procedure: Consider the set of all functions of one real variable. For $N \geq 2$, let the linear transformation L_N be defined by

$$L_N f(u) = \alpha_N(u)\, f[u\sqrt{N/(N - 1)} + b/\sqrt{(N - 1)}] + \beta_N(u)\, f[u\sqrt{N/(N - 1)} - a/\sqrt{(N - 1)}].$$

For example, if $f(u) = u$, and $\alpha_N(u) = \alpha$ (a constant), then

$$L_N f(u) = u\sqrt{N/(N - 1)} + (\alpha b - \beta a)/\sqrt{(N - 1)}.$$

Since

$$P_N(u) = L_N P_{N-1}(u),$$
$$P_N(u) = L_N L_{N-1} \ldots L_2 P_1(u),$$

where $P_1(u)$ is either zero or one, according to the value of u.

Let $\{g_N\}$ be an arbitrary sequence of functions, and let $h_N = P_N - g_N$. (In the sequel, g_N will be the approximation to P_N, so that h_N measures the difference between the exact solution of Eq. 14 and its approximation.) Let k be an arbitrary positive integer. From

$$h_k = P_k - g_k \quad \text{and} \quad P_{k+1} = L_{k+1} P_k,$$

it follows that

$$L_{k+1}\, h_k = P_{k+1} - L_{k+1}\, g_k\,;$$

and, since $P_{k+1} = h_{k+1} + g_{k+1}$, it follows that

$$h_{k+1} = L_{k+1}\, h_k - (g_{k+1} - L_{k+1}\, g_k).$$

Hence

$$h_{k+2} = L_{k+2}\, h_{k+1} - (g_{k+2} - L_{k+2}\, g_{k+1}),$$

or

$$h_{k+2} = L_{k+2}\, L_{k+1}\, h_k - (g_{k+2} - L_{k+2}\, g_{k+1}) - L_{k+2}\,(g_{k+1} - L_{k+1}\, g_k).$$

Repetition of this procedure gives rise to the following result:

$$h_N = L_N\, L_{N-1} \ldots L_{k+1}\, h_k - \sum_{j=k+1}^{N} R_j,$$

where

$$R_N = g_N - L_N\, g_{N-1}$$

and, for $k + 1 \leq j < N$

$$R_j = L_N \cdot \ldots \cdot L_{j+1}\,(g_j - L_j\, g_{j-1}).$$

Since α_j and β_j are positive functions, with sum one, it follows that, for any function f,

$$\operatorname*{lub}_{-\infty<u<\infty} |\, L_j\, f(u)\,| \leq \operatorname*{lub}_{-\infty<u<\infty} |\, f(u)\,|.$$

So $|\,R_j\,| \leq \operatorname*{lub}_{-\infty<u<\infty} |\, g_j(u) - L_j\, g_{j-1}(u)\,|$ for all $j = 2, 3, \ldots.$ Consequently, if

$$\sum_{j=2}^{\infty} \operatorname*{lub}_{-\infty<u<\infty} |\, g_j(u) - L_j\, g_{j-1}(u)\,|$$

converges, then

$$\sum_{j=k+1}^{N} R_j$$

can be made arbitrarily small simply by taking k sufficiently large, regardless of how large N may be.

Under such circumstances, it would follow that the sequence $\{h_N\}$ converges to zero [i.e., $g_N(u)$ is close to $P_N(u)$ for large N] provided that $L_N\, L_{N-1} \ldots L_{k+1}\, h_k$ converges to zero as N becomes infinite.

The preceding discussion may be summarized under two principles.

First Principle

In seeking a sequence $\{g_N\}$ to approximate $\{P_N\}$ try to satisfy the condition that $\sum_{j=2}^{\infty} \operatorname*{lub}_{-\infty<u<\infty} |\, g_j(u) - L_j\, g_{j-1}(u)\,|$ converges.

Second Principle

The sequence $\{P_N - g_N\}$ will converge to zero if the derived sequence

$$\{L_N\, L_{N-1} \ldots L_{k+1}\,(P_k - g_k)\} \quad (k \text{ fixed})$$

converges to zero as N becomes infinite.

it follows that

$$L_{k+1}\, h_k = P_{k+1} - L_{k+1}\, g_k\,;$$

and, since $P_{k+1} = h_{k+1} + g_{k+1}$, it follows that

$$h_{k+1} = L_{k+1}\, h_k - (g_{k+1} - L_{k+1}\, g_k).$$

Hence

$$h_{k+2} = L_{k+2}\, h_{k+1} - (g_{k+2} - L_{k+2}\, g_{k+1}),$$

or

$$h_{k+2} = L_{k+2}\, L_{k+1}\, h_k - (g_{k+2} - L_{k+2}\, g_{k+1}) - L_{k+2}\, (g_{k+1} - L_{k+1}\, g_k).$$

Repetition of this procedure gives rise to the following result:

$$h_N = L_N\, L_{N-1} \ldots L_{k+1}\, h_k - \sum_{j=k+1}^{N} R_j,$$

where

$$R_N = g_N - L_N\, g_{N-1}$$

and, for $k+1 \leq j < N$

$$R_j = L_N \cdot \ldots \cdot L_{j+1}\, (g_j - L_j\, g_{j-1}).$$

Since α_j and β_j are positive functions, with sum one, it follows that, for any function f,

$$\operatorname*{lub}_{-\infty < u < \infty} |\, L_j\, f(u) \,| \leq \operatorname*{lub}_{-\infty < u < \infty} |\, f(u) \,|.$$

So $|\, R_j \,| \leq \operatorname*{lub}_{-\infty<u<\infty} |\, g_j\,(u) - L_j\, g_{j-1}\,(u) \,|$ for all $j = 2, 3, \ldots.$ Consequently, if

$$\sum_{j=2}^{\infty} \operatorname*{lub}_{-\infty<u<\infty} |\, g_j\,(u) - L_j\, g_{j-1}\,(u) \,|$$

converges, then

$$\sum_{j=k+1}^{N} R_j$$

can be made arbitrarily small simply by taking k sufficiently large, regardless of how large N may be.

Under such circumstances, it would follow that the sequence $\{h_N\}$ converges to zero [i.e., $g_N\,(u)$ is close to $P_N(u)$ for large N] provided that $L_N\, L_{N-1} \ldots L_{k+1}\, h_k$ converges to zero as N becomes infinite.

The preceding discussion may be summarized under two principles.

First Principle

In seeking a sequence $\{g_N\}$ to approximate $\{P_N\}$ try to satisfy the condition that $\sum_{j=2}^{\infty} \operatorname*{lub}_{-\infty<u<\infty} |\, g_j\,(u) - L_j\, g_{j-1}\,(u) \,|$ converges.

Second Principle

The sequence $\{P_N - g_N\}$ will converge to zero if the derived sequence

$$\{L_N\, L_{N-1} \ldots L_{k+1}\, (P_k - g_k)\} \quad (k \text{ fixed})$$

converges to zero as N becomes infinite.

As noted earlier the conditions expressed in Eq. 9 imply that $\alpha_N(u)$ shall be a nondecreasing function of u. Before the two principles just enunciated can be applied, more information about the asymptotic behavior of $\alpha_N(u)$ is required. Throughout the rest of this section it will be assumed that

$$\begin{aligned} \alpha_N(u) &= p + (cu/\sqrt{N}) + \lambda(u,N)/N \\ \beta_N(u) &= q - (cu/\sqrt{N}) - \lambda(u,N)/N \end{aligned} \tag{14}$$

where p, q, and c are constants, with $p + q = 1$, and $\lambda(u, N)$ is bounded, for u in any finite closed interval. It might be remarked that for all the types of combat with which the author is familiar, Eq. 14 is satisfied.

A preview of the process that will presently be carried through is perhaps in order. First, a single function g of one real variable will be found that satisfies Eq. 13 except for terms of order $1/N^{3/2}$. Specifically, it is proposed to find a function g such that

$$g(u) - \alpha_N(u)\, g\{u\sqrt{N/(N-1)} + [b/\sqrt{(N-1)}]\} - \beta_N(u)\, g\{u\sqrt{N/(N-1)} - [a/\sqrt{(N-1)}]\} = 0\,(1/N^{3/2}). \tag{15}$$

This function, however, will not quite satisfy the boundary conditions demanded by Eq. 6.

Second, a sequence of functions $\{g_N(u)\}$ will be constructed from $g(u)$ that do satisfy the boundary conditions demanded by Eq. 6 and that still satisfy Eq. 13, except for terms of the order of $1/N^{3/2}$. Such a sequence will satisfy the conditions enunciated in the first principle.

Third, it will then be shown that Eqs. 9 and 14 imply that the conditions enunciated in the second principle are satisfied.

First Part

It is proposed to find a function g satisfying Eq. 15, which can be written (since $\alpha + \beta = 1$)

$$\alpha_N(u)\left(g\{u\sqrt{N/(N-1)} + [b/\sqrt{(N-1)}]\} - g(u)\right) + \beta_N(u)\left(g\{u\sqrt{N/(N-1)} - [a/\sqrt{(N-1)}]\} - g(u)\right) = 0\,(1/N^{3/2}).$$

Apply Taylor's theorem to the two differences that appear in this equation, and collect, separately, the terms involving $1/\sqrt{N}$ and $1/N$, ignoring those of higher order. The result is as follows:

$$(1/\sqrt{N})(pb - qa)\,g' + 1/2N\ \{[1 + 2(a+b)c]\,ug' + (pb^2 + qa^2)\,g''\} = 0\,(1/N^{3/2}).$$

It will be recalled that when the variable u was introduced it was stated that a and b were to be determined later. At this point it is possible to impose the first condition on a and b, namely, that the term involving $1/\sqrt{N}$ should vanish. This leads to the equation

$$pb - qa = 0; \tag{16}$$

consequently

$$\begin{aligned} a &= pk \\ b &= qk, \end{aligned}$$

when k is yet to be determined. It should be borne in mind that Eq. 16 is, in fact, an implicit equation, since the choice of a and b affects the values of p and

q in general. The specific example considered below will make this clear if it is not obvious already.

With $a = pk$ and $b = qk$, the term involving $1/N$ can be simplified; for $pb^2 + qa^2 = (pq^2 + qp^2)k^2 = pqk^2$. Then the requirement that the term in $1/N$ must also vanish leads to the following differential equation.

$$[1 + 2(a+b)c]\, ug'(u) + pqk^2 g''(u) = 0\,, \tag{17}$$

the solution of which is

$$g(u) = c_1 \int_{c_2}^{u} \exp\{-[1 + 2(a+b)c]t^2/(2pqk^2)\}\, dt \tag{18}$$

where c_1 and c_2 are arbitrary constants.

It should now be observed that the product $(a+b)c = kc$ is invariant, since c involves k only through a factor of $1/k$. This observation suggests the final determination for k. Simply choose k as $\sqrt{1+2(a+b)c/pq}$; the coefficient of $-t^2/2$ in the exponent of Eq. 18 now becomes unity. Thus, with these choices for a and b, it is seen that

$$g(u) = c_1 \int_{c_2}^{u} \exp(-t^2/2)\, dt. \tag{19}$$

Before proceeding further with the discussion of the general case it seems desirable to consider a specific example—one that arises from the classical Lanchester theory. Suppose that

$$\alpha(m,n) = \alpha m/(\alpha m + \beta n), \text{ and } \beta(m,n) = \beta n/(\alpha m + \beta n)\,, \tag{20}$$

where α and β now denote two positive constants. With $N = m+n$ and $u = (am - bn)/\sqrt{m+n}$), it is found that

$$\alpha_N(u) = p + c\,(u/\sqrt{N}) + \lambda(u,N)/N$$

and

$$\beta_N(u) = q - c\,(u/\sqrt{N}) - \lambda(u,N)/N\,,$$

in which

$$p = \alpha b/(\alpha b + \beta a)\,,$$

$$q = \beta a/(\alpha b + \beta a)\,,$$

and

$$c = \alpha\beta(a+b)/(\alpha b + \beta a)^2\,.$$

These expressions are found by substituting Eq. 11 in $\alpha m/(\alpha m + \beta n)$, and expanding in powers of $1/\sqrt{N}$. Equation 16 now becomes

$$\alpha b^2 - \beta a^2 = 0\,.$$

With a proper choice of signs it is seen that $a = k'\sqrt{\alpha}$, $b = k'\sqrt{\beta}$, where k' is a positive constant. This k' is not the same as the k in Eq. 17.

With these values for a and b it is noted that $p = \sqrt{\alpha}/(\sqrt{\alpha}+\sqrt{\beta})$, $q = \sqrt{\beta}/(\sqrt{\alpha}+\sqrt{\beta})$, and $(a+b)\,c = 1$. Now with $a = k[\sqrt{\alpha}/(\sqrt{\alpha}+\sqrt{\beta})]$, and $b = k\,[\sqrt{\beta}/(\sqrt{\alpha}+\sqrt{\beta})]$, Eq. 17 becomes

$$3ug'(u) + [\sqrt{\alpha\beta}/(\sqrt{\alpha}+\sqrt{\beta})]^2 k^2 g''(u) = 0.$$

The proper choice for k is, by the general theory, $\sqrt{[1+2(a+b)c]/pq}$, which here becomes $\sqrt{3}/\sqrt{\alpha\beta}$. Then $u=\sqrt{3}/\sqrt{\alpha\beta}\,[(\sqrt{\alpha}m-\sqrt{\beta}n)/\sqrt{m+n}]$, and Eq. 19 holds, i.e.,

$$g(u) = c_1 \int_{c_2}^{u} \exp(-t^2/2)\, dt.$$

Second Part

There now is left, in the general theory, the problem of choosing c_1 and c_2. There are actually many possible choices for c_1 and c_2, although they are all asymptotically equivalent. Choose, for example, c_1 as $1/\sqrt{2\pi}$ and c_2 as $-\infty$. It then turns out that $g(u)$ (now independent of N) satisfies the boundary condition along the coordinate axes only asymptotically. It is easy, however, by letting c_1 and c_2 depend on N, to obtain a sequence of functions $\{g_N(u)\}$ that actually satisfies the conditions that, if $n=0$, the value of $g_N(u)$ is 1, and, if $m=0$, the value of $g_N(u)$ is 0.

To see how this may be done consider the situation that occurs on the line $m+n=N$. At the point (0,N) on this line (where $P(0,N)=0$) u has the value $-b\sqrt{N}$. Since naturally c_1 is not to be 0, $g(u)$ will be 0 only if $c_2=-b\sqrt{N}$. Along this line $m+n=N$, u increases steadily until the point (N,0) is reached, at which $P(N,0)=1$. At (N,0), u has the value $a\sqrt{N}$. In order to obtain the proper value for $g(u)$, namely, 1, c_1 should be the reciprocal of

$$\int_{-b\sqrt{N}}^{a\sqrt{N}} e^{-\frac{1}{2}t^2}\, dt.$$

Thus, instead of simple constants for c_1 and c_2, values that depend on N are chosen. This leads to a consideration of the sequence of functions $\{g_N(u)\}$, where

$$g_N(u) = \left[\int_{-b\sqrt{N}}^{u} e^{-\frac{1}{2}t^2}\, dt\right] \Big/ \left[\int_{-b\sqrt{N}}^{a\sqrt{N}} e^{-\frac{1}{2}t^2}\, dt\right],$$

for

$$-b\sqrt{N} \le u \le a\sqrt{N},$$

and

$$g_N(u) = \begin{cases} 1 \text{ for } u \ge a\sqrt{N} \\ 0 \text{ for } u \le -b\sqrt{N}. \end{cases}$$

Because

$$\lim_{N\to\infty} N^k \int_{x\sqrt{N}}^{\infty} e^{-\frac{1}{2}t^2}\, dt = 0$$

for any positive numbers k and x, it follows that this sequence of functions also satisfies the fundamental difference equation for $P_N(u)$ to within terms of order $1/N$, just as did $g(u)$.

Third Part

According to the preview given earlier in this section all that remains is to prove that $L_N\, L_{N-1} \ldots L_{k+1}\, (P_k - g_k)$ approaches zero as N becomes infinite.

Let, as before, h_k denote $P_k - g_k$. Now h_k is a function of one real variable, defined at the values of u corresponding to the lattice points on the line $m+n=k$. Specifically, h_k is defined at all the numbers $[-bk + j(a+b)]/\sqrt{k}$, for $j = 0, \pm 1, \pm 2, \ldots$. However (and fortunately), it is zero at most of these points. In fact it is zero at every point except possibly the points $[-bk + j(a+b)]/\sqrt{k}$ for $j = 1, 2, \ldots, k-1$. This follows from the way in whick g_k was made to obey the boundary conditions of Eq. 6.

Let $D_{k,N}$ denote $L_N L_{N-1} \ldots L_{k+1} \; h_k$. Since $D_{k,N} = L_N D_{k,N-1}$, it is ar that $D_{k,N}$ is, for $N > k$, a solution of the fundamental difference equation (Eq. 3). Consequently the value of $D_{k,N}$ at any number u is a certain linear combination of the values of h_k, and since most of these are zero they can be written

$$D_{k,N}(u) = \sum_{j=1}^{k-1} F_{j,k,N}(u) \, h_k \left[\frac{-bk + j(a+b)}{\sqrt{k}}\right].$$

The point to be demonstrated is that $F_{j,k,N}(u)$ approaches zero as N becomes infinite.

As a first step it is proposed to show that, if

$$G_{k,N}(u) = \operatorname*{lub}_{(\text{all } j)} F_{j,k,N}(u),$$

then

$$G_{k,N}(u) \le G_{k+1,N}(u).$$

To this end observe that

$$\begin{aligned} F_{j,k,N}(u) = {} & \alpha_{k+1}\{[(a+b)j - b(k+1)]/\sqrt{k+1}\} \, F\{[(a+b)j - b(k+1)]/\sqrt{k+1}\} \\ & + \beta_{k+1}\{[(a+b)(j+1) - b(k+1)]/\sqrt{k+1}\} \, F\{[(a+b)(j+1) - b(k+1)]/\sqrt{k+1}\}, \end{aligned} \tag{21}$$

so that

$$F_{j,k,N}(u) \le \left(\alpha_{k+1}\{[(a+b)j - b(k+1)]/\sqrt{k+1}\} + \beta_{k+1}\{[(a+b)(i+1) - b(k+1)]/\sqrt{k+1}\}\right) G_{k+1,N}(u).$$

But, by Eq. 9 the coefficient of $G_{k+1,N}(u)$ is no greater than one. [If the coefficients here seem remote from Eq. 9 transform the coefficients here into $\alpha(j, k-j+1)$ and $\beta(j+1, k-j)$ by using Eq. 11; then apply Eq. 9.] Thus, for every j, $F_{j,k,N}(u) \le G_{k+1,N}(u)$, and from this, it follows that $G_{k,N}(u) \le G_{k+1,N}(u)$.

Next it is proposed to show that $G_{r,N}(u)$ can be made arbitrarily small, for fixed u, simply by taking r and $N-r$ sufficiently large.

It is obvious from Eq. 14 that $\alpha_r(u)$ will be sensibly constant on any bounded closed interval, if r is sufficiently large. Hence, as a little reflection will show, $G_{r,N}(u)$ will be near the maximum term in a binomial distribution involving $N-r$ trials with probability p of success in any one trial. But from the classical theorem of De Moivre and Laplace this is approximately $1/\sqrt{[2\pi(N-r)pq]}$, which is arbitrarily small if $N-r$ is sufficiently large. This shows that $G_{r,N}(u)$ can be made arbitrarily small by taking r and $N-r$ sufficiently large.

From the two stages of this discussion it now follows that $L_N L_{N-1} \ldots L_{k+1} \; (P_k - g_k)$ approaches zero as N becomes infinite.

The foregoing proof shows that $D_{k,N}$ approaches zero at least as fast as $1/\sqrt{N}$. It may in fact approach zero even faster, as the following example shows.

In the event that $\alpha(m,n)$ is not constant but actually increases with increasing m (and decreases with increasing n), the total contribution from any value of $P_k - g_k$ to all points on the line $m+n=k+1$ is less than one. Thus, if $\alpha(m,n) = m/(m+n)$,

the total contribution of any point on the line $m+n=k$ to all points on the line $m+n=k+1$ is $k/(k+1)$. By induction the total contribution of this point on $m+n=k$ to all points on $m+n=N$ is k/N. Consequently in this case D_{kN} approaches zero at least as fast as $1/N$.

Summary

In this section the probability that the first side wins in generalized Lanchester combat has been studied. With m combatants initially on the first side, and n combatants initially on the second side the probability that the first side wins is given approximately by

$$\left[\int_{-b\sqrt{m+n}}^{\frac{am-bn}{\sqrt{m+n}}} e^{-\frac{1}{2}t^2}\,dt\right] \Big/ \left[\int_{-b\sqrt{m+n}}^{a\sqrt{m+n}} e^{-\frac{1}{2}t^2}\,dt\right] \tag{22}$$

where a and b are constants to be determined by the relative effectivenesses of the two sides. A simpler form of the approximation, but which does not give quite so accurate results, is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{am-bn}{\sqrt{m+n}}} e^{-\frac{1}{2}t^2}\,dt. \tag{23}$$

In the particular case of the classical Lanchester theory, where $\alpha m/(\alpha m+\beta n)$ gives the probability that the next casualty will occur on the second side, it is seen that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{\frac{3}{\sqrt{\alpha}\sqrt{\beta}}}\,\frac{\sqrt{\alpha}m-\sqrt{\beta}n}{\sqrt{m+n}}} e^{-\frac{1}{2}t^2}\,dt, \tag{24}$$

or, more closely,

$$\left[\int_{-\sqrt{3\frac{\sqrt{\beta}}{\sqrt{\alpha}}}\,\sqrt{m+n}}^{\sqrt{\frac{3}{\sqrt{\alpha}\sqrt{\beta}}}\,\frac{\sqrt{\alpha}m-\sqrt{\beta}n}{\sqrt{m+n}}} e^{-\frac{1}{2}t^2}\,dt\right] \Big/ \left[\int_{-\sqrt{3\frac{\sqrt{\beta}}{\sqrt{\alpha}}}\,\sqrt{m+n}}^{\sqrt{3\frac{\sqrt{\alpha}}{\sqrt{\beta}}}\,\sqrt{m+n}} e^{-\frac{1}{2}t^2}\,dt\right] \tag{25}$$

approximates the probability that the first side wins.

CLASSICAL THEORY AND NUMERICAL EXAMPLES

This section is devoted to two somewhat separate topics. The first is the computation of $\phi(m,n)$ and $\alpha(m,n)$ for the situation described in the manuscript by Lee and Harrison.[16] It will be shown that the probabilities of casualties in the type of combat discussed there satisfy Eq. 20, so that the probability that Force 1 wins is given exactly by Eq. 8, and approximately by Eq. 24 or 25. The second topic is the comparison, in some numerical examples, of the exact and approximate probabilities in the unfavorable circumstances that both m and n are small.

The situation considered by Lee and Harrison is the following (with slight changes of notation):

Two forces are engaged in combat. Force 1 consists initially of m weapons, Force 2 consists initially of n weapons. Each weapon of Force 1 fires at random according to the Poisson process with a mean firing rate of λ shots per unit time. Each weapon of Force 2 fires with a mean firing rate of μ shots per unit time. The single-shot kill probability of a weapon on Force 1 firing at a weapon on Force 2 is p; the single-shot kill probability of a weapon on Force 2 firing at a weapon on Force 1 is q. The engagement is continued until all weapons of one force or the other are destroyed.

For a discussion of the Poisson process the reader is referred to Feller.[17] By the well-known properties of the Poisson process it follows that, if there are m combatants in Force 1 each firing at a rate of λ shots per minute, then the probability that Force 1 fires exactly j shots during a time interval of t minutes is

$$e^{-m\lambda t} \cdot [(m\lambda t)^j/j!] .$$

Since $1 - p$ is the probability that any one of these shots produces no casualty, the probability that all j shots miss is $(1 - p)^j$. Hence, the probability that Force 1 fires exactly j shots and all miss is

$$e^{-m\lambda t} \cdot [m\lambda(1-p)t]^j/j! .$$

Therefore the probability that Force 1 produces no casualties (regardless of the number of shots fired) during the time interval of t minutes is

$$\sum_{j=0}^{\infty} e^{-m\lambda t} \cdot [m\lambda(1-p)t]^j/j!,$$

or

$$e^{-m\lambda t} \cdot e^{m\lambda(1-p)t},$$

which equals $e^{-m\lambda pt}$.

Thus it has now been shown that, if there are m members in Force 1, the probability of no casualties among Force 2 during a time interval of length t

minutes is $e^{-m\lambda pt}$. In the same way if there are n members in Force 2 the probability of no casualties among Force 1 during a time interval of length t minutes is $e^{-n\mu qt}$.

Consequently the probability that, with m members on the first side and n members on the second side, there are no casualties on either side during a time interval of length t is $\exp[-(m\lambda p + n\mu q)t]$. Comparison of this with Eq. 2 shows that $\phi(m,n) = m\lambda p + n\mu q$.

The problem now is to compute $\alpha(m,n)$, which, it will be recalled from the first section, means the probability that if a casualty does occur it will occur on the second side.

The probability of no casualties among Force 1 and at least one casualty among Force 2 during an interval of t minutes is

$$e^{-n\mu qt}(1 - e^{-m\lambda pt}).$$

The derivative of this, for $t = 0$, is $m\lambda p$. But, according to the general theory developed in the first section, this must also be

$$\alpha(m,n) \cdot \phi(m,n).$$

Hence,

$$\alpha(m,n) = m\lambda p/(m\lambda p + n\mu q).$$

Similarly,

$$\beta(m,n) = n\mu q/(m\lambda p + n\mu q) .$$

In summary then it has been shown that $\alpha(m,n)$ and $\beta(m,n)$ satisfy Eq. 20, with the constants α and β equal to λp and μq, respectively. Therefore the probability of winning in this type of combat can be computed as the solution of Eq. 5, with the coefficients given by Eq. 20. The explicit solution given by Eq. 8 can be used, or, the probabilities can be obtained successively from Eq. 5 directly. This latter method is the one actually used in the manuscript of Lee and Harrison previously cited.

It is clear that the function given in Eq. 8 depends on the ratio of α to β, and not on their separate values. Let $C = \alpha/\beta$. For various values of C, the values of $P(m,n)$, rounded to three decimal places, are tabulated in the manuscript by Lee and Harrison. These values have been rounded to two decimal places and provide the basis for the "exact" figures in Tables 1 and 2. For Table 3, the "exact" data are computed directly from Eq. 5, with $C = 9$.

Tables 1, 2, and 3 provide a comparison between these exact values of $P(m,n)$ and the approximate values of $P(m,n)$ found by using Eq. 24 for $C = 1$, $C = 2$, and $C = 9$. The comparisons are made only in the most unfavorable circumstances, when both m and n are small. It is clear from the tables that, as soon as $m + n = 7$ or so, Eq. 24 should be applicable instead of Eq. 8 for all ordinary purposes of computation, provided that $1/9 \leq C \leq 9$. This agreement between the exact and approximate values of $P(m,n)$ seems to be surprisingly good.

In the tables the lower entry in each pair of lines is the exact value of $P(m,n)$, rounded to two decimal places. The upper entry in each square is the approximate value, computed by Eq. 24, also rounded to two decimal places.

Table 1

COMPARISON OF EXACT AND APPROXIMATE VALUES[a] OF $P(m,n)$ FOR $C = 1$

$m \backslash n$	0	1	2	3	4	5
0	–	0.04	0.01	0.00	0.00	0.00
	–	0.00	0.00	0.00	0.00	0.00
1	0.96	0.50	0.16	0.04	0.01	0.00
	1.00	0.50	0.17	0.04	0.01	0.00
2	0.99	0.84	0.50	0.22	0.08	0.02
	1.00	0.83	0.50	0.22	0.08	0.02
3	1.00	0.96	0.78	0.50	0.26	0.11
	1.00	0.96	0.78	0.50	0.26	0.11
4	1.00	0.99	0.92	0.74	0.50	0.28
	1.00	0.99	0.92	0.74	0.50	0.28
5	1.00	1.00	0.98	0.89	0.72	0.50
	1.00	1.00	0.98	0.89	0.72	0.50

[a]The upper figure in each pair of lines is the approximation, computed from Eq. 24. The lower figure is the exact value, computed from Eq. 8. All data are rounded to two decimal places.

Table 2

COMPARISON OF EXACT AND APPROXIMATE VALUES[a] OF $P(m,n)$ FOR $C = 2$

$m \backslash n$	0	1	2	3	4	5
0	–	0.06	0.01	0.00	0.00	0.00
	–	0.00	0.00	0.00	0.00	0.00
1	0.97	0.67	0.34	0.12	0.04	0.01
	1.00	0.67	0.33	0.13	0.04	0.01
2	1.00	0.94	0.72	0.47	0.25	0.12
	1.00	0.93	0.73	0.48	0.26	0.12
3	1.00	0.99	0.94	0.77	0.55	0.35
	1.00	0.99	0.93	0.78	0.57	0.37
4	1.00	1.00	0.98	0.93	0.80	0.63
	1.00	1.00	0.98	0.93	0.81	0.64
5	1.00	1.00	1.00	0.98	0.93	0.83
	1.00	1.00	1.00	0.98	0.93	0.83

[a]The upper figure in each pair of lines is the approximation, computed from Eq. 24. The lower figure is the exact value, computed from Eq. 8. All data are rounded to two decimal places.

Table 3

COMPARISON OF EXACT AND APPROXIMATE VALUES[a] OF $P(m,n)$ FOR $C = 9$

$m \backslash n$	0	1	2	3	4	5
0	–	0.16	0.08	0.04	0.02	0.01
	–	0.00	0.00	0.00	0.00	0.00
1	1.00	0.92	0.72	0.50	0.33	0.21
	1.00	0.90	0.74	0.41	0.29	0.18
2	1.00	1.00	0.98	0.91	0.79	0.65
	1.00	0.99	0.97	0.89	0.78	0.65
3	1.00	1.00	1.00	0.99	0.97	0.92
	1.00	1.00	1.00	0.99	0.96	0.91
4	1.00	1.00	1.00	1.00	1.00	0.99
	1.00	1.00	1.00	1.00	1.00	0.98
5	1.00	1.00	1.00	1.00	1.00	1.00
	1.00	1.00	1.00	1.00	1.00	1.00

[a]The upper figure in each pair of lines is the approximation, computed from Eq. 24. The lower figure is the exact value, computed from Eq. 8. All data are rounded to two decimal places.

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